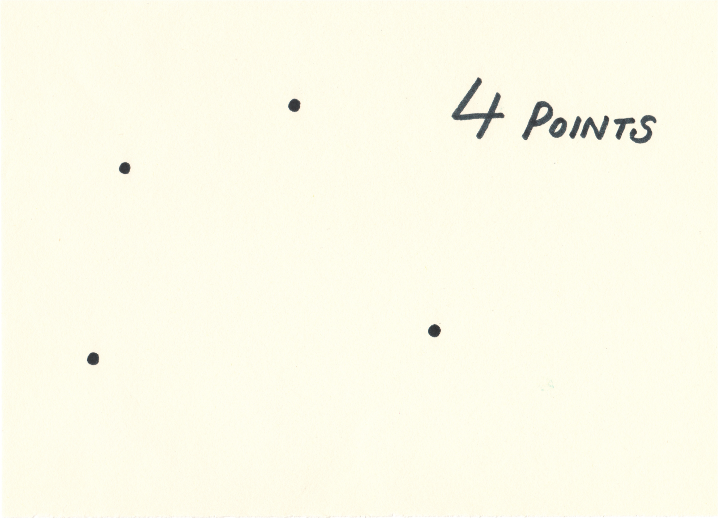
Points, lines, planes, etc.

June Huh

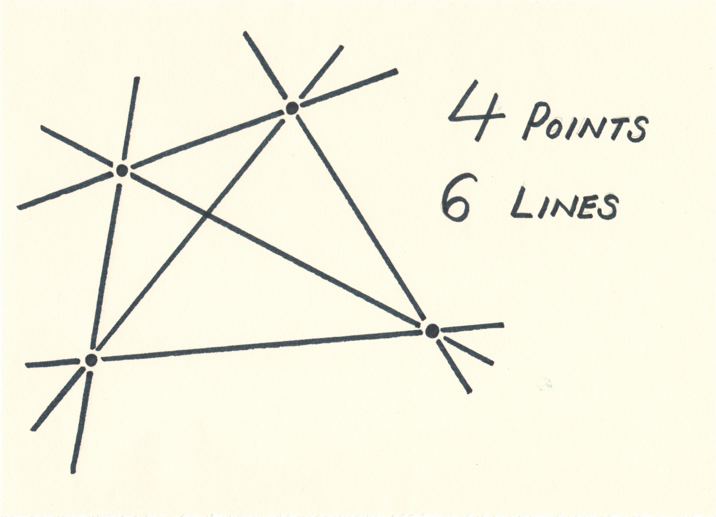
IAS

November 29, 2016

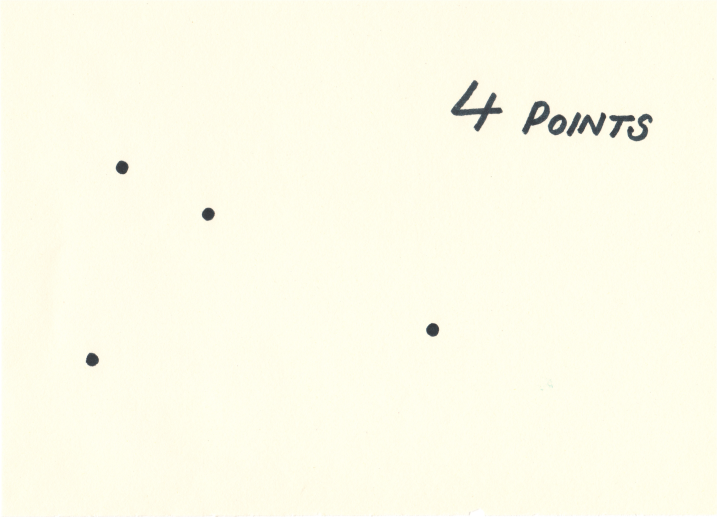
Here are 4 points in projective plane:



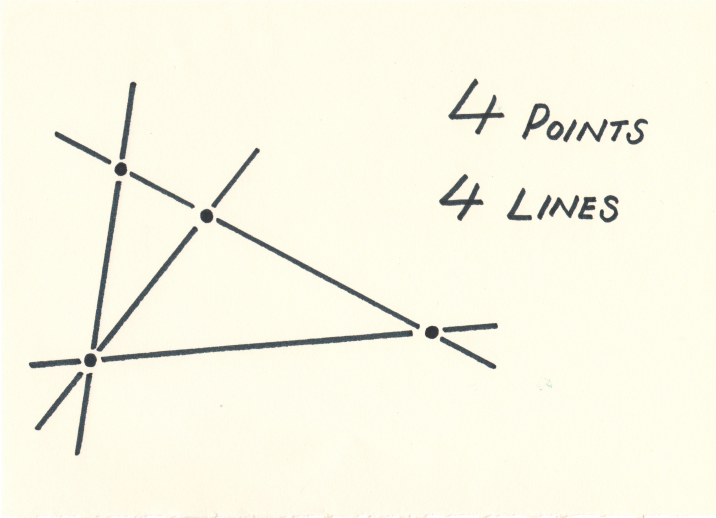
The 4 points determine 6 lines:



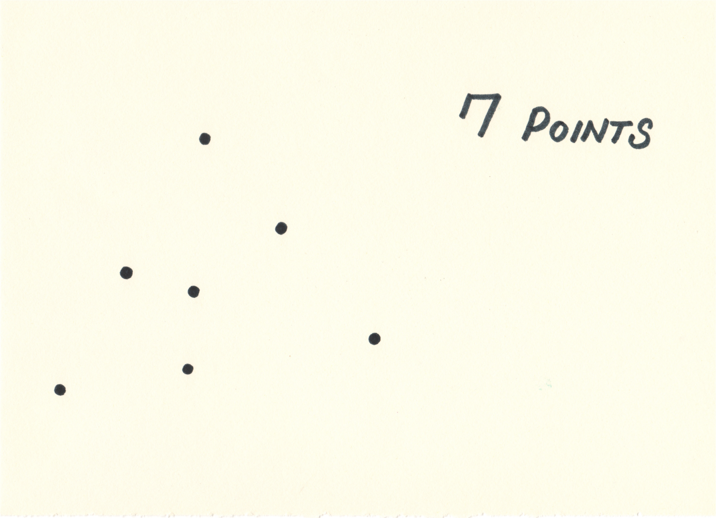
Let’s move one of the points into special position:



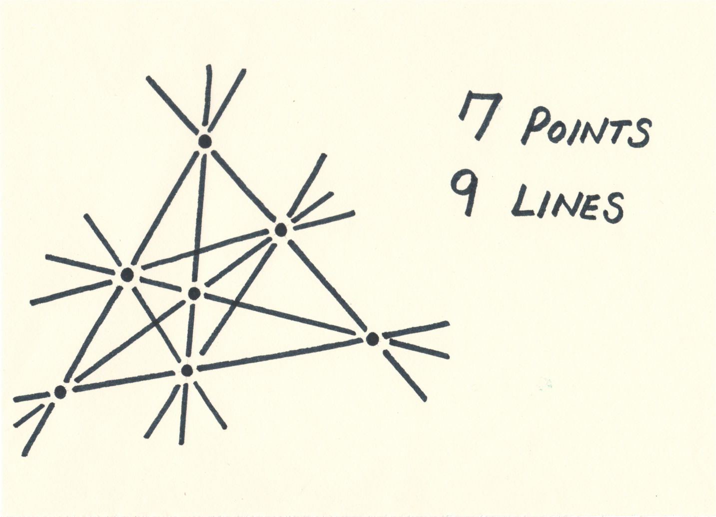
Now the 4 points determine 4 lines:



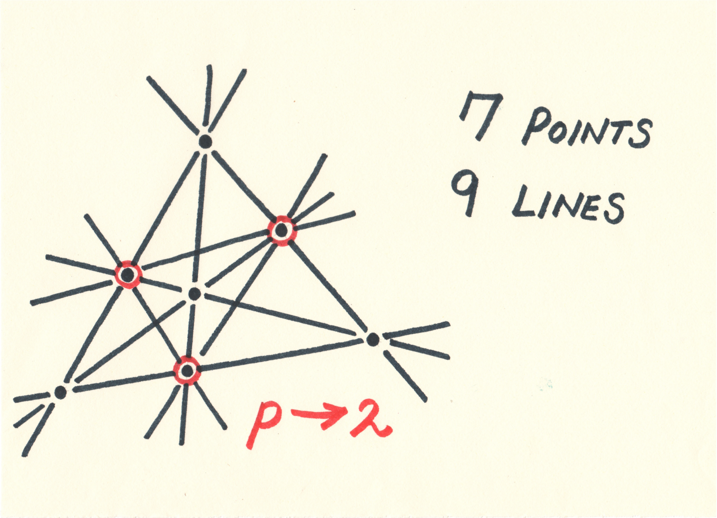
Think of all seven points with zero-one coordinates:



The 7 points determine 9 lines:



What happens if we move "p" in the base Spec$(\mathbb{Z})$ toward the prime number 2?



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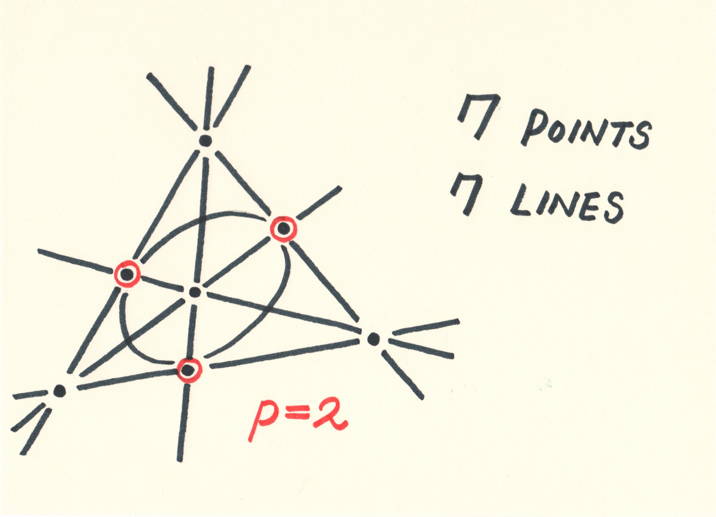


Theorem

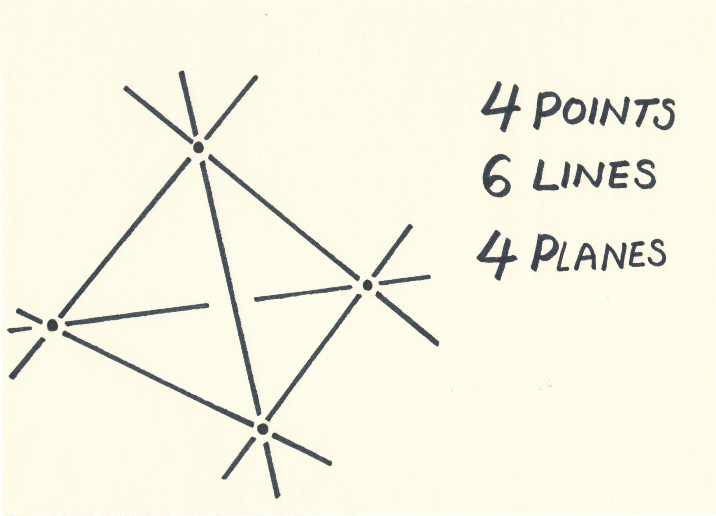
Every set of points E in a projective plane determines at least $|E|$ lines,

Theorem (de Bruijn and Erdős, 1948)

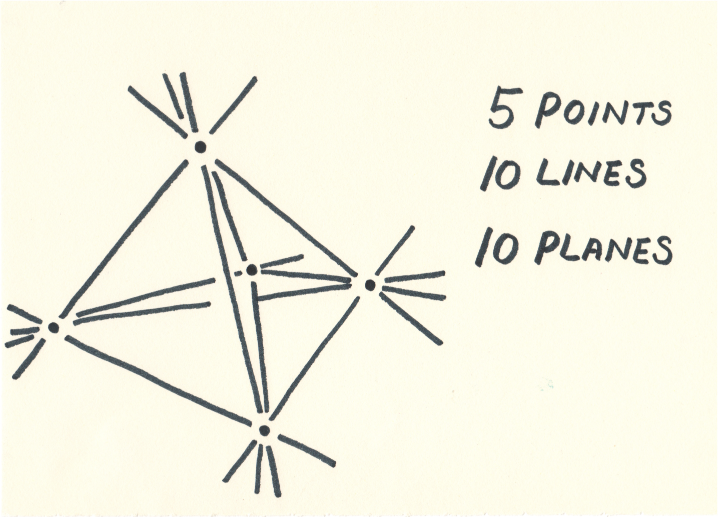
Every set of points E in a projective plane determines at least $|E|$ lines,

unless E is contained in a line.

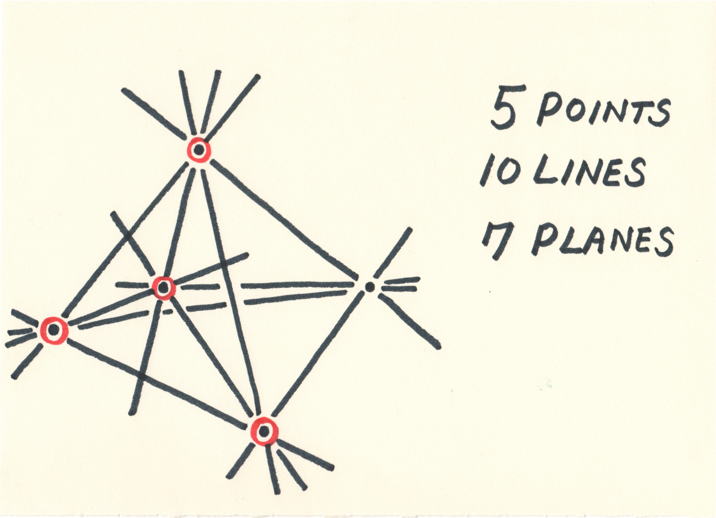
Here are 4 points in space defining 6 lines and 4 planes:



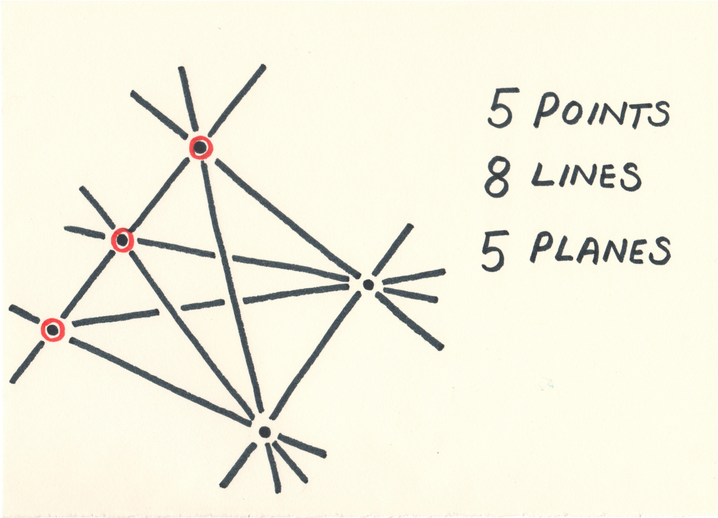
...and 5 points defining 10 lines and 10 planes:



. . . and 5 points defining 10 lines and 7 planes:



. . . and 5 points defining 8 lines and 5 planes:



Theorem (Motzkin, 1951)

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Motzkin worked over the real numbers and used properties of real numbers.

The assumption was removed by Basterfield and Kelly in 1968.

The theorem says that $|\mathscr{L}_1| \leq |\mathscr{L}_{d-1}|$.

Why is this true?

Theorem (Greene, 1970)

For every point (in $\mathscr{L}_1$) one can choose a hyperplane (in $\mathscr{L}_{d-1}$) containing the point so that no hyperplane is chosen twice (unless E is contained in a hyperplane).

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In other words, there is a *matching* from points to hyperplanes, where

$$\text{matching} = \left(\mathscr{L}_1 \xhookrightarrow{\iota} \mathscr{L}_{d-1}, \quad x \leq \iota(x) \ \text{ for all } x. \right)$$

Let E be a *spanning* subset of a d-dimensional vector space V, and

let $\mathscr{L}$ be the poset of subspaces of V spanned by subsets of E.

Write $\mathscr{L}_p$ for the set of p-dimensional spaces in $\mathscr{L}$.

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Example

If E is the set of 4 general vectors in $\mathbb{R}^3$ (4 *general points in projective plane*),

$$
\begin{aligned}
|\mathscr{L}_0| &= 1,\\
|\mathscr{L}_1| &= 4,\\
|\mathscr{L}_2| &= 6,\\
|\mathscr{L}_3| &= 1.
\end{aligned}
$$

Top-heavy Conjecture (Dowling and Wilson, 1974)

(1) *For every p less than $\frac{d}{2}$, we have*

$$|\mathscr{L}_p| \le |\mathscr{L}_{d-p}|.$$

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$$\mathscr{L}_0 \hookrightarrow \mathscr{L}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{L}_{d/2} \hookleftarrow \cdots \hookleftarrow \mathscr{L}_{d-1} \hookleftarrow \mathscr{L}_d.$$

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This implies *Sperner's theorem*:

"The maximum number of incomparable elements in the Boolean lattice is $\binom{d}{d/2}$."

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Canfield (1978) famously showed that the following stronger version of the conjecture (of Rota) fails for partition lattices:

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The smallest known partition lattice without the matching property has size $\geq 10^{10^{20}}$.

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For those who know about geometric lattices ($\simeq$ matroids):

The original formulation of the conjecture concerns arbitrary geometric lattices.

The current proof works only when $\mathscr{L}$ is "realizable" over some field.

We construct a graded commutative algebra

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The multiplication is defined by the rule

$$y_1 y_2 = \begin{cases} y_1 \vee y_2 & \text{if } \dim(y_1) + \dim(y_2) = \dim(y_1 \vee y_2), \\ 0 & \text{if } \dim(y_1) + \dim(y_2) > \dim(y_1 \vee y_2). \end{cases}$$

This is a graded version of the *Möbius algebra*.

We write L for the sum of all elements in $\mathscr{L}_1$, and show that

$$B^p(\mathscr{L}) \xhookrightarrow{L^{d-2p}} B^{d-p}(\mathscr{L}),$$

when $\mathscr{L}$ is realizable over some field. This will be enough.

We conjecture that the same holds without the assumption of realizability.

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(4) The permutation corresponding to this term produces the matching ι.

Okay, how to show that the linear map is injective?

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(4) The hard Lefschetz theorem for $IH^{2*}(f(X))$ gives the conclusion

$$B^p(\mathscr{L}) \xhookrightarrow{L^{d-2p}} B^{d-p}(\mathscr{L}).$$

In fact, after changing coefficients to $\mathbb{Q}_\ell$ if necessary, we have

$$B^*(\mathscr{L}) \simeq H^{2*}(f(X)).$$

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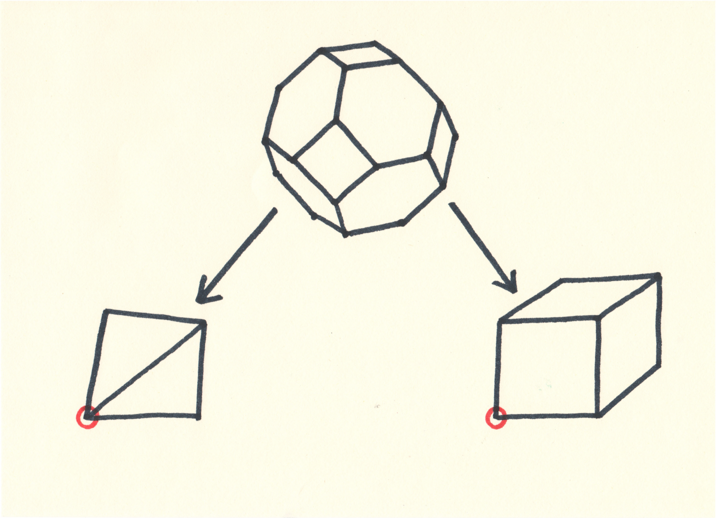
The construction of f for the Boolean lattice will induce all other constructions.

My favorite polytopes:



I will mark a vertex of the simplex and a vertex of the cube.

The corresponding toric varieties are

$$\begin{tikzcd}
 & X_{A_n} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^n & & (\mathbb{P}^1)^n \\
 & \mathbb{C}^n \arrow[ul, hook, "\iota_1"] \arrow[ur, hook', "\iota_2"'] &
\end{tikzcd}$$

The inclusions ι_1, ι_2 come from the markings.

The map π_1 is the blowup of all the torus invariant points in $\mathbb{P}^n$,

all the strict transforms of torus invariant $\mathbb{P}^1$'s in $\mathbb{P}^n$,

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The map π_2 is the blowup of points 0^n and ∞^n,

all the strict transforms of torus invariant $\mathbb{P}^1$'s in $(\mathbb{P}^1)^n$ containing 0^n or ∞^n,

all the strict transforms of torus invariant $(\mathbb{P}^1)^2$'s in $(\mathbb{P}^1)^n$ containing 0^n or ∞^n, etc.

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We use E to construct the inclusion $\iota_E : V^\vee \longrightarrow \mathbb{C}^n$.

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Consider the closure of $V^\vee$ in $\mathbb{P}^n$ and its strict transform

$$\begin{tikzcd} & X \arrow[r, hook, "j"] \arrow[dl, two heads] & X_{A_n} \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & \\ \overline{V}^\vee \arrow[r, hook, "\iota"] & \mathbb{P}^n & & (\mathbb{P}^1)^n. \end{tikzcd}$$

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\overline{V}^\vee \arrow[r, hook, "\iota"] & \mathbb{P}^n & & (\mathbb{P}^1)^n.
\end{tikzcd}$$

The map f we want is the composition $X \xrightarrow{j} X_{A_n} \xrightarrow{\pi_2} (\mathbb{P}^1)^n$.